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Modes of a Gaussian Random Walk

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It is demonstrated that a one-dimensional gaussian random walk (GRW) possesses an underlying structure in the form of random oscillatory modes. These modes are not sinusoids, but can be isolated by a well-defined procedure. They have average wavelengths and amplitudes, both of which can be determined by experiments or by theoretical calculations. This paper reports such determinations by both methods and also develops a theory that is ultimately shown to agree with experiments. Both theory and simulations show that the average wavelength and the average amplitude scale with the order of the mode in exactly the same way that the modes of the well-known Weierstrass fractal scale with mode order. This is remarkable since the wave generated by the Weierstrass function, $W(x) = \sum_{m=1}^{\infty} (\frac{1}{a})^m \cos(g^m x)$, is fully determined for the variable x whereas the GRW is stochastic. By increasing the size of the steps in the GRW, it is possible to selectively remove the fastest modes, while leaving the remaining modes almost unchanged. For a GRW, the parameters corresponding to a and g in the Weierstrass function are found to be 2.0 and 4.0, respectively. These values are independent of the variance associated with the GRW. Application of the random modes is reserved for a later paper.

KEY WORDS: Random walks; Brownian motion; probability theory; lattice statistical mechanics.

1. RANDOM WALK AS A FRACTAL

It has been recognized for some time that a random walk (RW) is a fractal.[1] Without characterizing it as such, Perrin in 1909[2] identified many of the features of the path of a Brownian particle as those that would identify a fractal. A well-known fundamental property of fractals is self-similarity or affine similarity that leads to scaling processes described by power laws.

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The purpose of this paper is to (i) demonstrate that there are well-defined random modes associated with a random process, (ii) to prescribe a method for finding these modes, and (iii) to describe some of the properties of such modes. We will focus on a gaussian random walk (GRW), a process that can be associated with Brownian motion. In later papers we will consider more general random processes. Self-similarity in a gaussian random walk is most simply described by "*coarse graining*" the walk, a simple renormalization procedure that falls under the heading of renormalization group (RG) theory, and which can obscure the finer details of the walk to any desired degree.

In a GRW the probability that a vector step lies in the volume element $d\mathbf{r}$ is given by

$$p(\mathbf{r}) = (2\pi\sigma_o^2)^{-d/2} \exp\left\{-\frac{(|\mathbf{r}^2|)}{2\sigma_o^2}\right\} \tag{1}$$

where d is the dimensionality of the walk and the constant σ_o^2 is its variance. One can coarse grain the walk by redefining the vector step as the sum of n original vector steps, i.e.,

$$\mathbf{r}' = \sum_{i=1}^{n} \mathbf{r}_i \tag{2}$$

where $\mathbf{r}_i$ represents the ith intermediate step. With the probability density of the ith intermediate step prescribed by Eq. (1), it is easy to show[3] that the probability for the volume element $d\mathbf{r}'$ is

$$p(\mathbf{r}') = (2\pi n\sigma_o^2)^{-d/2} \exp\left\{-\frac{(|\mathbf{r}'^2|)}{2n\sigma_o^2}\right\} \tag{3}$$

Writing σ^2 in place of $n\sigma_o^2$ converts Eq. (3) into an equation of the same form as Eq. (1), the only difference being the new variance. Thus the rescaled path of the GRW is self-similar to the original path. What has been lost is much information about the more finely scaled original path. Of course this is what RG theory usually does in an attempt to achieve convergence at a "*fixed point*,"[4] but this kind of suppression of detail is not our immediate interest. What we shall do is to prove that even a GRW contains distinct, discrete and well-defined oscillatory modes that scale with affine similarity.[5] It can be shown that any value of $n \geqslant 2$ in Eq. (2) preserves the gaussian distribution of the step sizes, but for $d = 1$, one particular value of n is favored ($n = 4$) because it removes the fastest vibrational mode from the random walk while at the same time leaving the frequencies

and amplitudes of all the remaining modes relatively unchanged. In this paper we will not attempt an application of these modes, but rather limit our consideration to the relevant scaling laws that are justified by experiment and some beginning theory. The modes are, of course, "*average*" quantities. They have average amplitudes and average wavelengths and it is these average quantities that satisfy the scaling laws. Thus the simple (almost obvious) scaling exemplified by Eqs. (1) through (3) can be accompanied by a much more detailed scaling that contains particular information about the GRW.

Of course a thoroughly random process can be Fourier analyzed in a straightforward manner into an infinite set of sinusoidal modes, but little understanding is gained from such a decomposition. The representation is purely formal since sinusoids have no natural connection with the process causing the random walk and so the oscillatory modes of interest to us will not be sinusoids. As a step towards understanding these modes, it is useful to examine what is believed to be the earliest formalization of a fractal, namely the Weierstrass series[5, 6] given by

$$W(x) = \sum_{m=1}^{\infty} \left(\frac{1}{a}\right)^m \cos(g^m x) \tag{4}$$

where x can be regarded as time or a length. If it is regarded as time, then g^m becomes an angular frequency ω_m. If it is regarded as a length, then $g^{-m} = \bar{\lambda}_m / 2\pi$ where $\bar{\lambda}_m$ is now a wavelength. Here a and g are positive integers greater than unity and $a = g^H$ with $0 < H < 1$. The Weierstrass function is remarkable in that it is continuous while not having a derivative anywhere. Fig. 1 plots a segment of the Weierstrass series for the case $g = 4.0$ and $H = 0.5$. It is to be noted that $W(x)$ is a fully deterministic function of x although a rough glance at Fig. 1 may give the opposite impression. To emphasize this last point we plot, in Fig. 2, a segment of a simulated one-dimensional GRW in which the ordinate is the one-dimensional position of the walker and the abscissa is a length measured in the number of steps, i.e., the length of the walk. The plot is for a GRW with standard deviation equal to 0.0005. This choice of standard deviation is not important and it is not related to the choice of $g = 4.0$ and $H = 0.5$ in Eq. (4). These particular values of g and H come about, below, from our analysis of the oscillatory modes of the GRW.

What is important is the striking similarity between the plots in the two figures, one of a curve based on a deterministic formula, and the other, a plot of a curve that is completely random. In the next section, we describe the procedure for finding the above-mentioned modes.

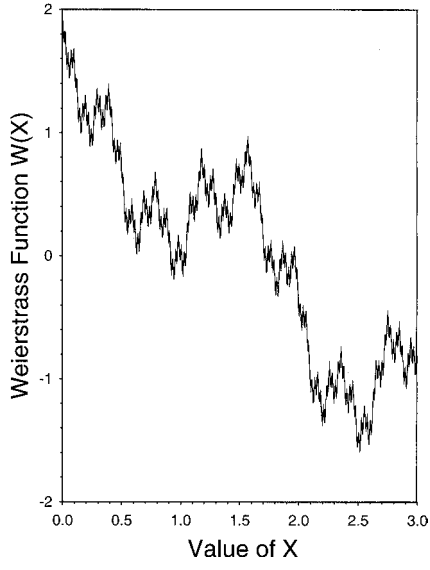

Fig. 1. Plot of the first 40 terms of the Weierstrass function (Eq. (4)) with $a = 2.0$ and $b = 4.0$ plotted over the range $x = 0.0$ to 3.0.

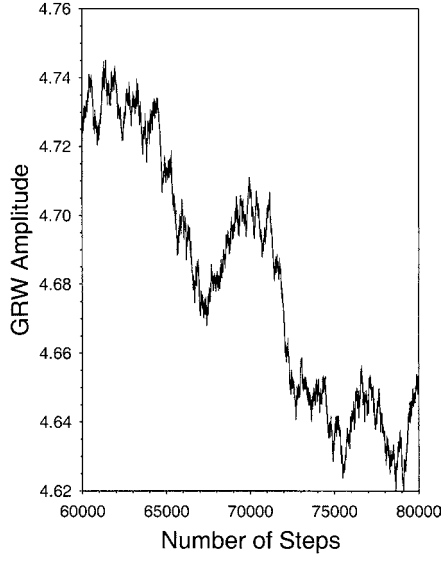

Fig. 2. Plot of a portion of a GRW simulation with average step size = 0.0 and standard deviation = 0.0005 over the range 60000 steps to 80000 steps.

2. ISOLATING THE MODES

Consider a one dimensional RW of infinite length. We can generate such a walk on a computer using a table of random numbers to determine the lengths of the individual steps. The total number of steps completed by the walker at a given time can be taken as a measure of that time, i.e., the time consumed by a step is constant. The RW can be plotted, as in Fig. 2, showing the position of the walker as the ordinate and the number of steps, or the time, as the abscissa. This sort of plot will show peaks and valleys, where a peak is always followed by a valley and vice versa. Every peak will be defined to be a "*first order*" peak and every valley, defined to be a "*first order*" valley. These peaks and valleys will be denoted by $+1$ and -1, respectively. The distance between two successive peaks will be regarded as a wavelength in much the same way as the distance between two crests of a sinusoidal wave may be defined as the wavelength. Alternatively, the distance between two valleys may be regarded as a wavelength in much the same way as the distance between two troughs of a sinusoid is a wavelength. Of course, the wavelength of a monochromatic sinusoid is a well-defined constant, whereas, in a particular random walk, the distance between two peaks (or between two valleys) will vary considerably.

On the other hand a random walk will exhibit an average first order wavelength. As a special example, consider a RW consisting of steps with the probability $1/2$ of being in the forward direction and $1/2$ of being in the backward direction. Then it is easy (see below) to show that the average distance between peaks will be 4 steps, so that the average wavelength is 4.

Now we will generate a set of "*second order*" peaks by a coarse graining process. We examine all the peaks in a RW and identify those that are higher than their nearest neighbor peaks. This is the set of second order peaks. We can identify second order valleys in the similar way (they are lower than their nearest neighbor valleys). We can then plot the walk showing only second order peaks and valleys (a walk more coarse grained than the first order walk). This second order walk will also exhibit an average wavelength, namely the average distance between two peaks, which will also be same as the average distance between two valleys. In the second order process there are some exceptions that will have to be noted such as the possibility that two successive second order peaks may not be separated by a second order valley. However, in this first description of the method we will ignore such complications. Also, in the case of constant and finite step length, a modest departure from the GRW definition, several first order peaks in succession may have exactly the same height while the sequence is bounded on both sides by lower peaks. The second order peaks and valleys will be denoted by $+2$ and -2, respectively.

Third order peaks and valleys can be determined using the second order peaks and valleys. Fourth order peaks and valleys can then be determined using the third order objects. The fifth order can be determined from the fourth, and so on. Clearly, the process of coarse graining can in principle be continued in this way indefinitely. Only if the random walk path contains a finite number of steps, will there be an upper limit on the "*wave order*" for a particular RW.

In order to demonstrate this process, we need an example of a RW. It is convenient to obtain such an "*experimental*" example from exhaustive simulations of such walks in which the step probabilities are determined by various random number generators. We have also made use of published tables of random numbers obtained from the decay of radioactive elements to generate experimental random walks. Alternatively, one could attempt a theoretical analysis that would yield the properties of the random modes. In this paper, we develop such a rigorous theoretical analysis for a special random walk consisting of steps where each one has the same amplitude. Only the direction (i.e., up or down) varies randomly. This is not quite a GRW, but in the appropriate limit, it is. We have conducted simulations of walks having constant step amplitude so that our theoretical analysis could be compared with an exact experimental example. Many of the conclusions derived from the case with constant step length are applicable to a true GRW (i.e., the existence of random modes) and have been confirmed experimentally in GRW's as well. In several of the following sections of this paper we present the theory for the constant step length walk. We compare theory and experiment. Also certain scaling features of the walk are compared with similar features of the Weierstrass series. Such a comparison of features is also made for the simulated true GRW and the Weierstrass function.

3. EVALUATION OF THE WAVELENGTH OF THE FIRST ORDER RANDOM MODE

As we have indicated, in order to clarify the above process of coarse graining, it is useful to perform an exact probability analysis on a random walk that is simpler than a GRW, by replacing the gaussian distributed length of each step with a length that is constant. Only the sign of the step's length will be allowed to vary. We will ignore, for the time being, the case where two second or higher order peaks are not separated by an intervening valley and the similar case of two valleys not having an intervening peak. A more immediate problem, caused by each step having an equal length, is that neighboring peaks or valleys may (as we mentioned earlier) also have identical heights or depths. For a GRW, where the step length is

a continuous random variable, peaks with exactly the same height form a set of measure zero. However, in some situations already mentioned, because of practical limits on experimental measurement in a true GRW or because we deal explicitly with a walk having a constant step size, it is possible and even probable that neighboring peaks (or valleys) may have exactly the same height. For instance, we must deal with this problem when a sequence of peaks have the same height and are higher than the peak that immediately precedes the sequence as well as the peak that immediately follows it. In this case, the entire sequence is treated as a single high order peak. A similar treatment is accorded to a sequence of valleys of equal depth that are immediately preceded and immediately followed by valleys that are of lesser depth.

In this section we will evaluate the wavelength of a first order random mode. It is convenient to begin our analysis with an unsymmetrical walk in which the probability of a forward step is p_1 and that of a backward is q_1. Later, we shall set $p_1 = q_1 = 1/2$, so as to obtain the results for a symmetrical walk. In order to identify a first order peak we must have a run of at least one backward step, followed by a run of n forward steps ($n \geqslant 1$), and finally followed by a run of at least one backward step. The probability of this arrangement is $q_1 p_1^n q_1$ and since the peak can be composed of any number of steps, we must sum over the probabilities of all possible n. Then the probability of a first order peak is

$$p^{(1)} = \sum_{n=1}^{N-2} q_1 p_1^n q_1 = q_1^2 \sum_{n=1}^{N-2} p_1^n = q_1^2 \left[\frac{p_1 - p_1^{N-1}}{1 - p_1} \right] \tag{5}$$

where N is the number of steps in the walk. For the limit $N \to \infty$, this becomes

$$p^{(1)} = \frac{q_1^2 p_1}{1 - p_1} \tag{6}$$

Eq. (6) is also the fraction of steps that mark the beginning of a peak (i.e., the fraction of steps that form a valley) and its reciprocal must be the average distance between peaks or the average "*wavelength.*" Thus, since the distance between peaks corresponds to a wavelength, for the symmetrical walk with $p_1 = q_1 = 1/2$, the average wavelength is

$$\lambda_1 = \frac{1 - p_1}{q_1^2 p_1} = \frac{1 - 1/2}{(1/2)^3} = 4 \tag{7}$$

a result mentioned earlier. The analysis for the second order average wavelength is somewhat more involved, but still simple. The method is described in the following section.

4. EVALUATION OF THE WAVELENGTH OF THE SECOND ORDER RANDOM MODE.

We need to know the probability that a $+1$ peak, immediately following another $+1$ peak, is higher than this previous peak, as well as the probability that a $+1$ peak, immediately following another $+1$ peak, is lower than the previous peak. Both of these probabilities are dependent on the probability that a $+1$ peak immediately following another $+1$ peak has exactly the same height. From the probabilities of these three outcomes, it is possible to calculate the probability of any second order peak and hence, its frequency in the random walk. A $+1$ peak can be formed by a single downward step at the end of a run of upward steps. There may be more than one downward step, but there must be at least one. Let m represent the number of downward steps in excess of the one that is necessary to form a peak. The next peak is formed by n upward steps, with $n > 0$, followed by a single downward step. This is illustrated in Fig. 3. Every path that forms a peak following another peak will be a special case of Fig. 3. So, in general, the probability of any path where a peak follows another peak is equal to

$$p_1^n q_1^{m+1} \tag{8}$$

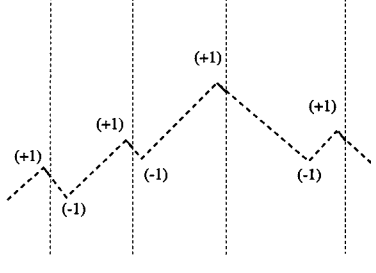

Fig. 3. Plot of a segment of a random walk showing a series of $+1$ peaks. In the downward sequence that forms each peak, the first step has been drawn as a solid line. The remainder of each downward sequence plus the upward sequence that follows it has been drawn with dashes since its lengths is variable. The vertical lines mark the location one step past each peak in the random walk and three complete first order cycles are displayed.

where the values of m and n depend on the specific path. The quantity Q_{1e}, which we use to represent the probability for the case where the following $+1$ peak is the same height as the previous $+1$ peak, has the value

$$Q_{1e} = \sum_{m=0}^{\infty} p_1^{m+1} q_1^{m+1} = \frac{(1-p_1)(1-q_1)}{1-p_1 q_1} \tag{9}$$

The quantity Q_{1+}, which we use to denote the probability for the case where the following $+1$ peak is higher than the previous $+1$ peak, is equal to

$$Q_{1+} = \sum_{m=0}^{\infty} \sum_{n=m+2}^{\infty} p_1^n q_1^{m+1} = \left[\frac{p_1}{1-p_1} \right] Q_{1e} \tag{10}$$

Finally, the quantity Q_{1-}, which we define to be the probability for the case where the following $+1$ peak is lower than the previous $+1$ peak, is calculated to be

$$Q_{1-} = \sum_{m=1}^{\infty} \sum_{n=1}^{m} p_1^n q_1^{m+1} = \left[\frac{q_1}{1-q_1} \right] Q_{1e} \tag{11}$$

The sum of Eqs. (9), (10) and (11) represents the probability of all possible sequences of two peaks and is therefore equal to unity, independent of the specific value of p_1 (or q_1), as it should be.

From the definition of a second order peak given in Section 2, it follows that the probability of a second order peak is the product $Q_{1+} Q_{1-}$. This represents a sequence of three $+1$ peaks with the first and third peaks both lower than the middle peak. This is one type of second order peak, but, according to our earlier definition, it is not the only possibility. Other types involve the middle peak repeating one or more times with exactly the same height. We denote the total probability of all possible second order peaks as b_2^{-1}, and it follows that

$$b_2^{-1} = Q_{1+} Q_{1-} \left[\sum_{m=0}^{\infty} Q_{1e}^m \right] = Q_{1+} Q_{1-} \left[\frac{1}{1-Q_{1e}} \right] \tag{12}$$

For $p_1 = q_1 = 1/2$, $Q_{1e} = Q_{1+} = Q_{1-} = 1/3$ and $b_2^{-1} = 1/6$. This means that, on average, one peak out of every six $+1$ peaks is a $+2$ peak (i.e., $\lambda_2 = 6$ first order peaks). Then, since a first order peak occurs on the average every four steps, a second order peak will occur on the average every 24 steps (i.e., $\lambda_2 = 24$ steps).

5. EVALUATION OF THE WAVELENGTH OF THE THIRD ORDER RANDOM MODE

Third order peak wavelength is determined by the probability of a second order peak being either higher or lower than the previous second order peak. These latter two probabilities can be calculated in the following manner. The probability that a $+1$ peak is one step higher than the previous $+1$ peak is $p_1 Q_{1e}$. The probability that a $+1$ peak is two steps higher than the previous $+1$ peak is $p_1^2 Q_{1e}$. Adding the probabilities of this form for all possible heights obviously gives Eq. (10). Replacing p_1 with q_1 gives the corresponding probabilities for a succeeding $+1$ peak that is lower and summing all the declining peak probabilities gives Eq. (11). Now, consider a "*staircase*" sequence of three succeeding $+1$ peaks, with the third peak being equal to or greater in height than the second peak and the second peak being equal to or greater in height than the first peak. The probability for a sequence like this has the form $p_1^s Q_{1e}^2$ where s is difference in height between the last (third) peak and the first peak. For a similar sequence of four $+1$ peaks, the probability has the form $p_1^s Q_{1e}^3$ where s is again the difference in height between the last (fourth) peak and the first peak. The general probability for a sequence of $+1$ peaks is

$$pp(s, t) = x_1^s Q_{1e}^t \tag{13}$$

where $s \geqslant 1$ is the total height change over the sequence and $t+1$ is the total number of peaks in the sequence. When $x_1 = p_1$, Eq. (13) applies to sequences with increasing peak heights and when $x_1 = q_1$, Eq. (13) applies to sequences with decreasing peak heights.

Eq. (13) must still be multiplied by a combinatorial factor to convert it into a more useful probability. This combinatorial factor allows for the fact that in general the s excess steps may be distributed over the peaks in several different ways and that each such distribution (having the same number of peaks and the same number of excess steps) is equally likely. Let the combinatorial factor be represented by $f(s, t)$, then

$$f(s, t) = \sum_{n=0}^{s} f(n, t-1) = \frac{(s+t-1)!}{s!(t-1)!} \tag{14}$$

where $s \geqslant 0$ and $t \geqslant 1$. Indeed Eq. (14) is the combinatorial function associated with Bose-Einstein statistics.[7] The expression $[1/(1-x)]^t$ constitutes a generating function for $f(s, t)$, and

$$\sum_{s=0}^{\infty} f(s, t)\, x^s = \frac{1}{(1-t)!} \frac{d^{t-1}}{dx^{t-1}} \left[\frac{x^{t-1}}{1-x} \right] = \left[\frac{1}{1-x} \right]^t \tag{15}$$

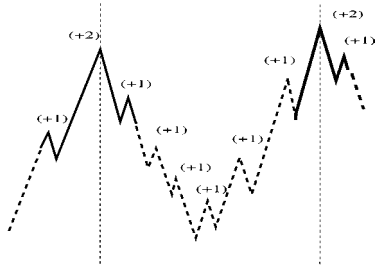

Fig. 4. Plot of a segment from the path of a random walk showing a +2 peak following another +2 peak. The dashed part of the path is variable. The portion of the path between the vertical lines represents one complete 2nd. order cycle.

for $t \geqslant 1$. Using this relationship, the probabilities for the occurrence of a sequence of $t+1$ increasing peaks of any height and $t+1$ decreasing peaks of any height are found to be, respectively, $(Q_{1+} + Q_{1e})^t$ and $(Q_{1-} + Q_{1e})^t$.

The segment of a random walk path proceeding from the top of a +2 peak to the top of the next +2 peak can have one of several forms (see Fig. 4). By definition, the first +2 top in the segment must be followed by at least one +1 top that is lower in height. There may be a sequence of several +1 tops following this +2 top, each one lower than the previous peak, but there must be at least one +1 top following the +2 top. In order to have a following +2 peak, the declining sequence of +1 peaks must be followed by at least one +1 peak that is higher than the last peak in the declining sequence. Instead of only one increasing +1 peak, there could be a sequence of +1 peaks, each one higher than the preceding peak, with the first peak higher than the last peak in the declining sequence. The simplest path between the two +2 peaks is, of course, one lower +1 peak followed by one higher +1 peak. The general probability for the occurrence of a segment of any type proceeding from a +2 peak to a succeeding +2 peak can be written in the form

$$Q_{1-}(Q_{1-} + Q_{1e})^m \, Q_{1+}(Q_{1+} + Q_{1e})^n \tag{16}$$

The structure represented by this equation should be clear. The terms in parentheses are the probabilities of the declining and increasing sequences of +1 peaks, containing $m+1$ and $n+1$ peaks, respectively. The initial Q_{1-} factor is the probability that the first+2 peak is indeed followed by a lower +1 peak. The Q_{1+} factor is the probability that the second+2 peak is higher than the lowest +1 peak. Thus the product of the probabilities in Eq. (16) is just the probability for the occurrence of a segment of any type proceeding from a +2 peak to the succeeding +2 peak, i.e., it is the probability that we are seeking. The probability of the simplest or minimum

path has $m = n = 0$, which is the probability that a $+1$ peak is followed by a lower $+1$ peak multiplied by the probability that a $+1$ peak is followed by a higher $+1$ peak. This path has the fewest number of peaks, not necessarily the fewest number of steps. In general, the downward portion of the wave will consist of a sequence of $m+1$ first order peaks where each peak is equal to or lower in height than the previous one and $m \geqslant 0$. The upward portion of the path consists of a sequence of $n+1$ first order peaks where each one is equal to or higher than the previous peak and $n \geqslant 0$. The end of the upward portion of the path is a second order peak, since it is assumed to be followed immediately by a downward segment in the random walk (which is not included in Eq. (16)). Eq. (16) is very similar to Eq. (8). Summing over all values of m and n (which here represent the number of peaks) in Eq. (16) is unity, as it should be, since any particular segment is certain to be one of these segments between two $+2$ peaks (for the first order equivalents, see Eqs. (9), (10), and (11)).

It is convenient to collect all the terms in Eq. (16) that correspond to the same difference of heights between the $+2$ peaks. This leads to the quantity

$$sp_{1-}(m)\, sp_{1+}(n) \tag{17}$$

where $sp_{1-}(m)$ is the sum of the probabilities of all sequences of decreasing $+1$ peaks with a total height decrease of m steps and $sp_{1+}(n)$ is the sum of the probabilities of all sequences of increasing $+1$ peaks with a total height increase of n steps. The sp_{1-} expression comes from the two terms on the left of Eq. (16) and the sp_{1+} expression comes from the two terms on the right of Eq. (16). Thus the probability of a random walk segment with a sequence of decreasing $+1$ peaks and a total height decrease of m steps, followed by a sequence of increasing $+1$ peaks and a total height increase of n steps, is given by Eq. (17). Here the net change in height from one $+2$ peak to the next is $n-m$ steps. The two quantities sp_{1+} and sp_{1-} in Eq. (17) are calculated from the following expression

$$sp_1(s) = x_1^s Q_{1e} \left\{ 1 + \sum_{k=1}^{s} f(k, 1) \left(\sum_{t=1}^{\infty} f(s-k, t)\, Q_{1e}^t \right) \right\} \tag{18}$$

for $s \geqslant 1$. When $x_1 = p_1$, Eq. (18) is equal to $sp_{1+}(s)$ and when $x_1 = q_1$, Eq. (18) is equal to $sp_{1-}(s)$. The first term inside the brackets of Eq. (18) (i.e., unity) covers the case where there are only two peaks in the segment and all the s excess steps are located on the second peak. The double sums in Eq. (18) cover the cases where there are three or more peaks in the segment, with k excess steps on the second peak and $s-k$ excess steps

distributed on the remaining peak(s). The following relation is useful in evaluating Eq. (18)

$$\sum_{t=1}^{\infty} f(s,t)\, x^t = \frac{x}{s!}\frac{d^s}{dx^s}\left[\sum_{t=1}^{\infty} x^{t+s-1}\right] = x\left[\frac{1}{1-x}\right]^{s+1} \tag{19}$$

for $s \geqslant 0$. We then obtain the relationships

$$sp_{1+}(s) = p_1^s\left[\frac{1}{1-Q_{1e}}\right]^s Q_{1e} \tag{20}$$

$$sp_{1-}(s) = q_1^s\left[\frac{1}{1-Q_{1e}}\right]^s Q_{1e} \tag{21}$$

We are now in a position to evaluate Q_{2e}, the probability that a $+2$ peak has the same height as the $+2$ peak immediately preceding it

$$Q_{2e} = \sum_{m=1}^{\infty} sp_{1-}(m)\, sp_{1+}(m) = \frac{(1-p_2)(1-q_2)}{1-p_2q_2} \tag{22}$$

where

$$p_2 = p_1\left[\frac{1}{1-Q_{1e}}\right] \tag{23}$$

$$q_2 = q_1\left[\frac{1}{1-Q_{1e}}\right] \tag{24}$$

$$(1-p_2)(1-q_2) = p_2q_2Q_{1e}^2 \tag{25}$$

If by Q_{2+} we represent the probability that the immediately succeeding second order peak is higher than the preceding second order peak, then

$$Q_{2+} = \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} sp_{1-}(m)\, sp_{1+}(n) = \left[\frac{p_2}{1-p_2}\right] Q_{2e} \tag{26}$$

In this case the increasing part of the segment is always higher than the decreasing part and the shortest declining segment has a height of minus one step. Finally, let the symbol Q_{2-} represent the probability that the immediately succeeding second order peak is lower than the preceding second order peak

$$Q_{2-} = \sum_{m=2}^{\infty}\sum_{n=1}^{m-1} sp_{1-}(m)\, sp_{1+}(n) = \left[\frac{q_2}{1-q_2}\right] Q_{2e} \tag{27}$$

where the increasing part of the segment is always shorter than the decreasing part (i.e., $n < m$).

Notice that Eqs. (22), (26), and (27) are very similar to Eqs. (9) through (11) in Section 3. The functional form of the two sets of equations is the same; only the magnitudes of the variables are altered slightly. This is the source of the fractal self-similarity of the $+1$ and $+2$ waves in the walk. The fraction of second order wave tops that are also third order tops is derived in exactly the same way as the fraction in Eq. (12), but using Q_{2e}, Q_{2+} and Q_{2-} instead of Q_{1e}, Q_{1+} and Q_{1-}. We find this fraction to be

$$b_3^{-1} = Q_{2+} Q_{2-} \left[\frac{1}{1 - Q_{2e}} \right] \tag{28}$$

For $p_1 = q_1 = 1/2$, the ratios of $1/7$, $3/7$ and $3/7$ are the respective magnitudes for Q_{2e}, Q_{2+} and Q_{2-} and $b_3^{-1} = 21/98$. So on average, one out of every 4.667 second order peaks is a third order peak (i.e., $\lambda_3 = 4.667$ second order peaks or 112.0 steps).

6. EVALUATION OF THE WAVELENGTHS OF THE HIGHER ORDER RANDOM MODES

The evaluation of the fraction of third order peaks that are also fourth order peaks requires the exact repetition of the process carried out in Section 5. Only the nature of the $p's$ and $q's$ needs to be changed. Likewise the fraction of fourth order waves that are also fifth order waves, repeats exactly the calculations of Section 5, again with only the nature of the $p's$ and $q's$ changing. This is true for all higher order waves. Eqs. (29) through (34) give an exact description of this repeating pattern in terms of the wave order parameter, n.

$$p_n = p_{n-1} \left[\frac{1}{1 - Q_{n-1e}} \right] \tag{29}$$

$$q_n = q_{n-1} \left[\frac{1}{1 - Q_{n-1e}} \right] \tag{30}$$

$$Q_{ne} = \frac{(1 - p_n)(1 - q_n)}{1 - p_n q_n} \tag{31}$$

$$Q_{n+} = \left[\frac{p_n}{1 - p_n} \right] Q_{ne} \tag{32}$$

$$Q_{n-} = \left[\frac{q_n}{1 - q_n} \right] Q_{ne} \tag{33}$$

for $n \geqslant 1$. Eqs. (29) through (33) are the generalizations of Eqs. (9) through (11), as well as Eqs. (22), (26), and (27). The general expression for the b factor is

$$b_{n+1}^{-1} = Q_{n+} Q_{n-} \left[\frac{1}{1-Q_{ne}} \right] = \lambda_{n+1}^{-1} \tag{34}$$

for $n \geqslant 1$. For the special case of a random walk with $p_1 = q_1 = 1/2$, the following relationships hold

$$p_n = 1-(1/2)^n \tag{35}$$

$$q_n = 1-(1/2)^n \tag{36}$$

7. ANALYSIS OF RESULTS

It is useful to tabulate some of the quantities appearing in the preceding analytical theory for given values of $p_1 = 1-q_1$. In particular, the symmetrical case, $p_1 = q_1 = 1/2$, is of interest. Table I presents such a tabulation for this case. As for experiments, we have performed exhaustive simulations of random walks; both true gaussian walks and walks having a constant step length. At each step, the step's direction and (for a GRW) the step's length were determined by an appropriate random number generator. Several different generators were used, and all led to the same overall result. In addition, the random number generator was replaced by published data on radioactive decay. This also led to the same result.

For generating uniform pseudo-random numbers between 0.0 and 1.0 on a computer, the experiments described in the tables of this paper used

Table I. Listing of the Calculated Values of Eq. (29) Through (34) for Random Wave Orders from $n=1$ to 8 Where $p_1 = q_1 = 1/2$. The b Factors Are Dimensionless Ratios and the Rest of the Entries Are Probabilities

n	p_n	q_n	Q_{ne}	Q_{n+}	Q_{n-}	b_{n+1}
1	1/2	1/2	1/3	1/3	1/3	6
2	3/4	3/4	1/7	3/7	3/7	98/21
3	7/8	7/8	1/15	7/15	7/15	210/49
4	15/16	15/16	1/31	15/31	15/31	930/225
5	31/32	31/32	1/63	31/63	31/63	3906/961
6	63/64	63/64	1/127	63/127	63/127	16002/3969
7	127/128	127/128	1/255	127/255	127/255	64770/16129
8	255/256	255/256	1/511	255/511	255/511	260610/65025

an algorithm called the Mersenne Twister.[8] This uniform distribution was then converted to a gaussian distributed random variable for the GRW experiments using an algorithm by D. Knuth.[9,10] We also used uniform random numbers determined experimentally from a radioactive decay process which were published on an internet site by J. Walker.[11]

Table II shows a comparison between values of b_n, evaluated theoretically, using Eqs. (29) through (34), and values obtained experimentally (via a computer simulation with the Mersenne Twister). The results were averaged over 2570 separate walks each of which contained over 25 million individual steps. The values are for a walk having a constant step length and $p_1 = q_1 = 1/2$. Quantities are listed out to the eighth order random wave. Clearly, the agreement between theory and experiment is excellent. Moreover, we see that, as the wave order increases, b_n converges on the constant 4.0. Thus, confirming that as the wave order is increased, this scaling property of the random modes becomes identical with that of the Weierstrass series modes.

This raises the question of why b_n is different for low order waves. The answer may be discerned, at least partially, from an examination of the simulations for true GRW's where the step length is not constant. In these simulations $b_n = 4$, independent of the wave order. In the case of constant step length, coarse graining increases with increasing wave order and the probability of two succeeding waves having exactly the same height decreases. Ultimately, a highly coarse grained wave will relegate a small constant step length to the continuum. The probability of two waves having exactly the same height will approach zero and the phenomenon will approach that of a true GRW. Table III compares the theoretical and experimental wavelengths, measured in units of step length. Again, the agreement between theory and experiment is excellent.

Table II. Comparison of the Theoretical and Experimental Values of b_n for Random Wave Orders from $n = 2$ to 8 Where $p_1 = q_1 = 1/2$. The b Factors Are, of Course, Dimensionless Ratios

b factor	Calculated	Experimental
b_2	6.00000	5.99991
b_3	4.66667	4.66667
b_4	4.28571	4.28583
b_5	4.13333	4.13318
b_6	4.06452	4.06537
b_7	4.03175	4.03402
b_8	4.01575	4.02440

Table III. Comparison of the Theoretical and Experimental Values of λ_n for Random Wave Orders $n=1$ to 8 Where $p_1 = q_1 = 1/2$. The Wave Lengths Are Expressed in Units of Steps

n	Calculated wavelength	Experimental wavelength
1	4.0	4.0000
2	24.0	23.9996
3	112.0	111.9997
4	480.0	479.9900
5	1984.0	1984.3260
6	8064.0	8065.3169
7	32512.0	32543.8887
8	130560.0	131110.6758

The wave amplitude can be defined as the height of a peak measured from the bottom of the preceding valley. However, even for the walk of constant step length, we do not yet have a complete analytical theory for the average amplitude, beyond a result for the first order wave. On the other hand, some experimental values are available. Table IV lists experimental amplitudes, in units of step length, out to the eighth order wave, as well as the amplitude ratios of successive waves. The single analytical result for the first order wave amplitude is 2.0, and for that case, agreement between experiment and theory is perfect. At this point we can return to Eq. (4) in which $\frac{1}{a} = g^{-H}$ where $0 < H < 1$. Thus the amplitude $(\frac{1}{a})^m$ of the mth mode in Eq. (4) is given by

$$\bar{A}_m = (b^{-H})^m \tag{37}$$

In view of the similarity of the scaling properties of the *wavelengths* of the Weierstrass and random modes, it now becomes of interest to see whether, also, the *amplitudes* of the random modes will scale like those of the Weierstrass modes. In a study of this question, we are limited to the experimental data since no analytical theory yet exists for the amplitude. Examination of the data in Table IV shows that the experimental amplitude goes as $A_n \approx 2^n$, at least as far as the higher order waves are concerned. This relationship could be expressed as

$$A = (b^H)^n \tag{38}$$

with $b = 4.0$ and $H = 1/2$ (parameters that would also fit the demands of the Weierstrass modes). With these values for b and H, the quantity in the

Table IV. Listing of Experimental Amplitudes and Experimental Amplitude Ratios for Random Wave Orders $n=1$ to 8 Where $p_1 = q_1 = 1/2$. The Wave Amplitude Is Expressed in Units of Steps. The Wave Amplitude Ratios Are Dimensionless

n	Experimental wave amp.	Experimental amp. ratio
1	2.0000	
2	5.7878	2.8939
3	13.4617	2.3259
4	28.8187	2.1408
5	59.5283	2.0675
6	120.9639	2.0320
7	243.6485	2.0142
8	489.6018	2.0095

parentheses of Eq. (38) is 2 as Table IV requires. Of course, the correspondence is only approximate for the lower order modes (as was the case for the wavelength), but again, Table IV refers to a walk with constant step length and the simulative data for the GRW is free of this defect. For a GRW the first order wave amplitude is 2.0 and the amplitude for each succeeding mode is twice the amount of the previous mode's amplitude. This result has been verified experimentally.

It has been mentioned that when scaling a GRW according to Eq. (2), certain values of the parameter n are preferred because they remove one or more of the fastest modes while leaving the remaining random modes largely unaffected. This is demonstrated in Table V where the results of random mode analysis are shown for a GRW containing 1.2 million steps. The same GRW was analyzed with $n = 1$, 4 and 16 (i.e., taking every step, every fourth step and every sixteenth step in the GRW). The top entry in each column is the value of n. The next entry down is the number of first order cycles in the scaled GRW, the entry just below that is the number of second order cycles, and so on. It is easy to see that changing n from 1 to 4 removes the 299031 cycles of the first order random mode in the GRW (with $n = 1$ and $\lambda_1 = 4$ steps). With the larger step size there are now 75178 cycles for the first order random mode ($n = 4$ and $\lambda_1 = 4n$ steps) which is essentially the same as the number of second order waves in the original GRW ($n = 1$ and $\lambda_2 = 16$ steps). Increasing n from 4 to 16 removes this mode as well, but still leaves the remaining random modes essentially unchanged. Thus this GRW is systematically "*coarse grained*" in the manner of RN theory.

Table V. Listing of the Number of Random Waves of All Orders Found in a 1.2 Million Step GRW Using Three Values of the Scaling Parameter *n* from Eq. (2). The Top Entry in Each Column Is the Value of *n*. The First Non-Blank Entry Below *n* Is the Number of First Order Waves Found in the GRW. The Second Non-Blank Entry Is the Number of Second Order Waves Found in the GRW, etc. The Table Illustrates How for Certain Preferred Values of *n*, the Increasing Step Size Eliminates the Fastest Random Mode(s) While at the same Time Leaving the Remaining Random Modes Almost Unchanged

$n=1$	$n=4$	$n=16$
299031		
74469	75178	
18489	18722	18761
4626	4657	4677
1153	1154	1170
291	294	300
69	69	69
20	18	17
6	5	5

In summary, we have demonstrated that there are well defined random modes associated with a random walk. We have given a simple method for identifying these and we have derived their structure and properties from the basic mathematics governing a random walk. We have shown that the scaling properties of the fundamental modes of a GRW are identical to those of a Weierstrass mode. To a degree, this is remarkable, especially since the path of the Weierstrass function is *fully determined* while that of the GRW is *random*. It should be indicated that there exists a considerable literature[12] on the relation between a gaussian-like process and a Weierstrass-like function, but the identification and quantification of the random modes discussed in this paper does not seem to have been accomplished. There are several possible applications of such fundamental random modes and these can be developed in following papers. Also, it is of interest to see if such modes can be found in quasi-random processes, like the stock market, in which correlation may play a role. We leave these matters to later papers.

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